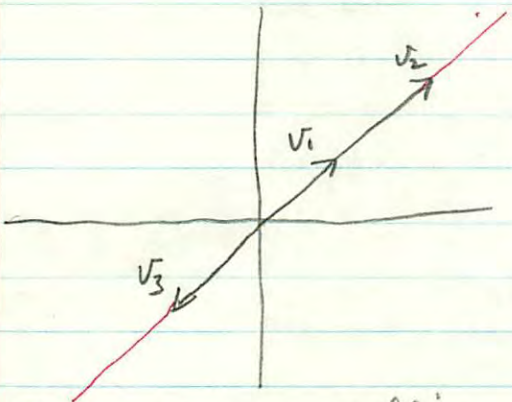
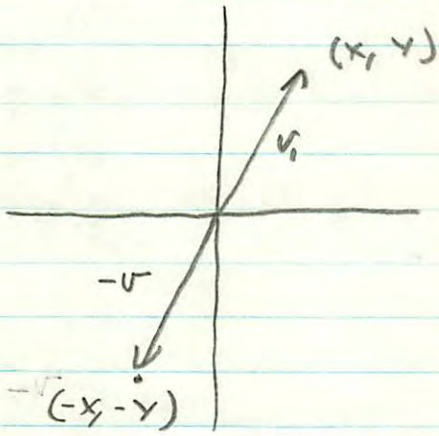
a two dimensional vector is written $\vec{v} = (x, y)$
Initially we will consider a vector in terms of its Geometric representation.

a Geometric vector corresponding to $\vec{v} = (x, y)$ is in standard position if it begins at $(0,0)$ and ends at (x, y). If it begins at (x_1, y_1) and ends at (x_2, y_2) it is still a vector. a vector has a "norm" written $\|v\|$. a Geometric vector has a "magnitude". In either case we will understand that to be the real number associated with v; which is its length.
Thus if $v = (x, y)$ then $\|v\| = \sqrt{x^2 + y^2}$.

a vector has a "direction angle". a Geometric vector has a direction. the angle is measured counterclockwise from horizontal. $\cos\theta = \frac{x}{\|v\|}$
$\sin\theta = \frac{y}{\|v\|}$ $0 \le \theta < 360°$ Note: $\tan\theta = \frac{y}{x}$ can't tell which is negative using this method.
Two vectors are said to be equal if their direction angle and norm are equal. If $v_1 \& v_2$ are in standard position then they would be coincident



v_1, v_2, v_3 are collinear since they lie on the same line.



the negative of $v = (x, y)$ is $-v = (-x, -y)$.

the zero vector has 0 length and is omnidirectional. that is it has any direction which is convenient.

Basic operations w/ Geometric Vectors
Have Students follow through examples beginning page 211.

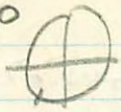
① The Sum of two vectors (also called the resultant of v_1 & v_2.

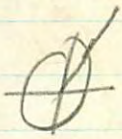
② Scalar product. a Scalar is a real number. cv is the vector collinear with v which has length $|c|\|v\|$. $v=(x,y)$ then $cv=(cx,cy)$
if $c>0$ then cv has the same direction as v.
if $c<0$ then cv is in the opposite direction.

① $(-2,-2)$

$\|(-2,-2)\| = \sqrt{4+4} = 2\sqrt{2}$ $\cos\theta = \frac{-2}{2\sqrt{2}} = -\frac{\sqrt{2}}{2}$

$\sin\theta = -\frac{\sqrt{2}}{2}$ $\therefore$ $\theta = 225°$





② $(1,\sqrt{3})$ $\sqrt{1+3} = 2$ $\cos\theta = \frac{1}{2}$ $\sin\theta = \frac{\sqrt{3}}{2}$ $\theta = 60°$

③ $(-3,0)$ $\sqrt{(-3)^2+0^2} = 3$ $\cos\theta = -1$ $\sin\theta = 0$ $\theta = 180°$

④ $(0,6)$ 6 $\cos\theta = 0$ $\sin\theta = 1$ $\theta = 90°$

⑤ $(\sqrt{3},-1)$ $\|v\| = 2$ $\cos\theta = \frac{\sqrt{3}}{2}$ $\sin\theta = -\frac{1}{2}$ $\theta = 330°$

⑥ $\left(-\frac{\sqrt{3}}{2}, \frac{7}{2}\right)$ $\|v\| = \sqrt{\frac{147}{4} + \frac{49}{4}} = \sqrt{\frac{196}{4}} = \sqrt{49} = 7$

$\sin\theta = \frac{\frac{7}{2}}{7} = \frac{1}{2}$ $\cos\theta = \frac{-\frac{2\sqrt{3}}{2}}{7} = -\frac{\sqrt{3}}{2}$

$\theta = 150°$

⑦ $(-3,3)$ $\|v\| = \sqrt{4+9} = \sqrt{13}$ requires calculator.

⑧ calculator

⑨ $\|v\| = 2$ $\theta = 60$ $\cos 60 = \frac{x}{2}$ $\sin 60 = \frac{y}{2}$

$\frac{1}{2} = \frac{x}{2}$ $\frac{\sqrt{3}}{2} = \frac{y}{2}$

$x = 1$ $\sqrt{3} = y$

$v = (1,\sqrt{3})$.

⑩ $\|v\| = \sqrt{2}$ $\theta = 45°$ $\cos 45 = \frac{x}{\sqrt{2}}$ $\sin 45 = \frac{y}{\sqrt{2}}$

$\frac{\sqrt{2}}{2} = \frac{x}{\sqrt{2}}$ $1 = y$

$v = (1,1)$ $1 = x$

(11) $\|v\| = 4 \quad \theta = 180$ $\quad \sin 180 = \frac{y}{4}$ $\quad \cos 180 = \frac{x}{4}$

$v = (-4, 0)$ $\quad y = 0$ $\quad x = -4$

(12) $\|v\| = 3 \quad \theta = 270^\circ$ $\quad \sin 270^\circ = \frac{y}{3}$

$(0, -3)$ $\quad y = -3$

(13) $\|v\| = 4 \quad \theta = 120^\circ$ $\quad \cos 120^\circ = \frac{x}{4}$ $\quad \sin 120^\circ = \frac{y}{4}$

$(-2, 2\sqrt{3})$ $\quad -\frac{1}{2} = \frac{x}{4}$ $\quad \frac{\sqrt{3}}{2} = \frac{y}{4}$

(14) $\|v\| = 8 \quad \theta = 225^\circ$ $\quad \cos 225^\circ = \frac{x}{8}$

$(-4\sqrt{2}, -4\sqrt{2})$ $\quad -\frac{\sqrt{2}}{2} = \frac{x}{8}$

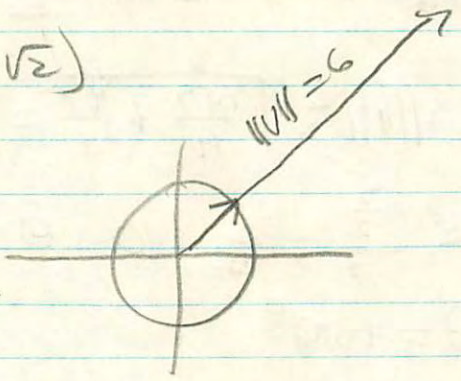
(15) $3(\sqrt{2}, \sqrt{2}) = (3\sqrt{2}, 3\sqrt{2})$

$\|v\| = \sqrt{18 + 18} = 6$

$\cos\theta = \frac{3\sqrt{2}}{6} = \frac{\sqrt{2}}{2}$

$\sin\theta = \frac{3\sqrt{2}}{6} = \frac{\sqrt{2}}{2}$

$\theta = 45^\circ$



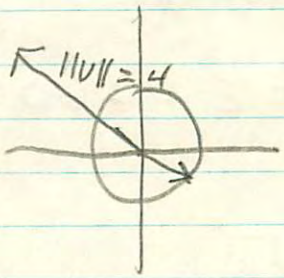
(16) $-2(\sqrt{3}, -1) = (-2\sqrt{3}, 2)$

$\sqrt{3 + 1} = 2$ $\quad \sqrt{12 + 4} = 4$

$\cos\theta = \frac{\sqrt{3}}{2}$ $\quad \cos\theta = -\frac{\sqrt{3}}{2}$

$\sin\theta = \frac{1}{2}$ $\quad \sin\theta = \frac{1}{2}$

$\theta = 150^\circ$



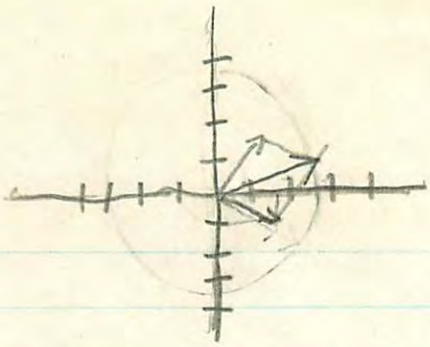
(17) $u_1 + u_2$

$(\sqrt{3}, -1)$ $(\sqrt{2}, \sqrt{2})$

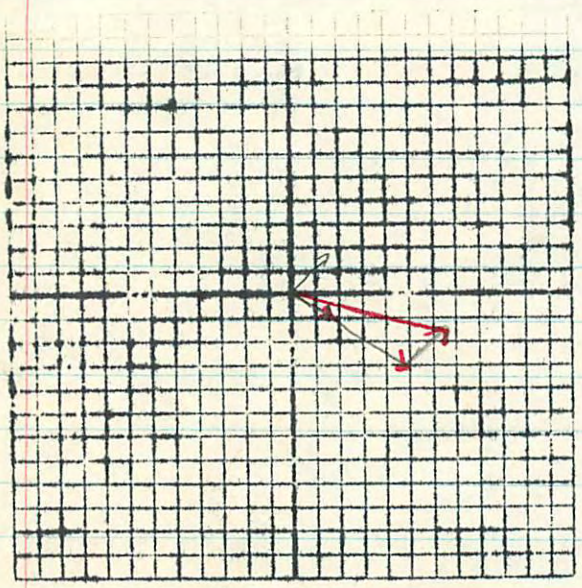
$\cos\theta = \frac{\sqrt{3}}{2}$

$\sin\theta = \frac{1}{2}$

$(\sqrt{3}+\sqrt{2}, -1+\sqrt{2})$

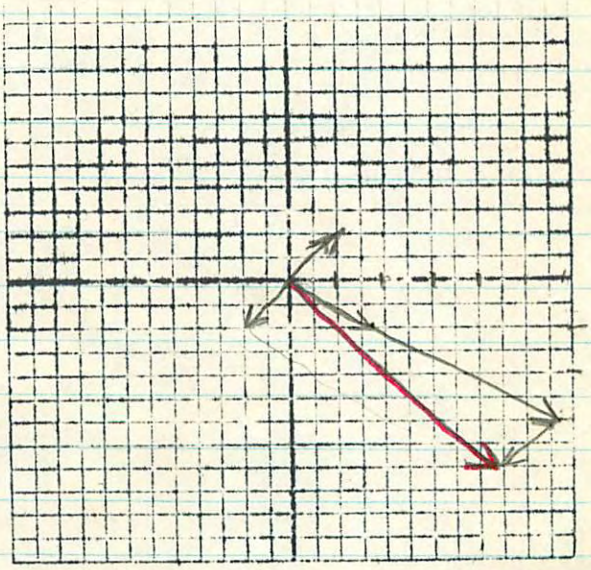


(18) $3(\sqrt{3}, -1) + (\sqrt{2}, \sqrt{2})$

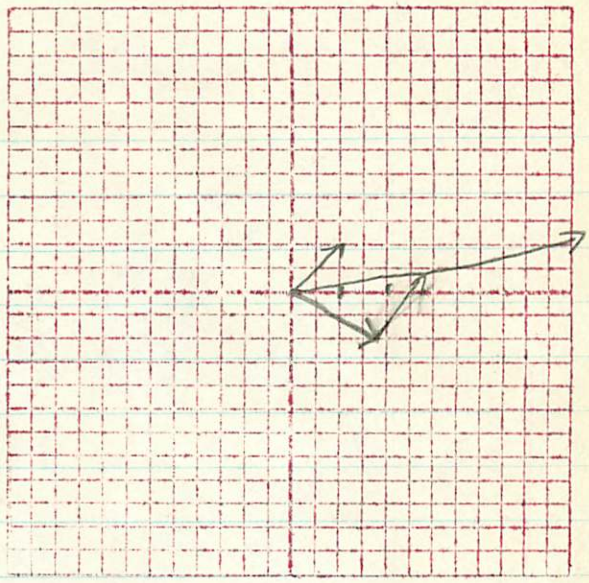


$(3\sqrt{3}+\sqrt{2}, -3+\sqrt{2})$

(19) $3(\sqrt{3}, -1) - (\sqrt{2}, \sqrt{2})$

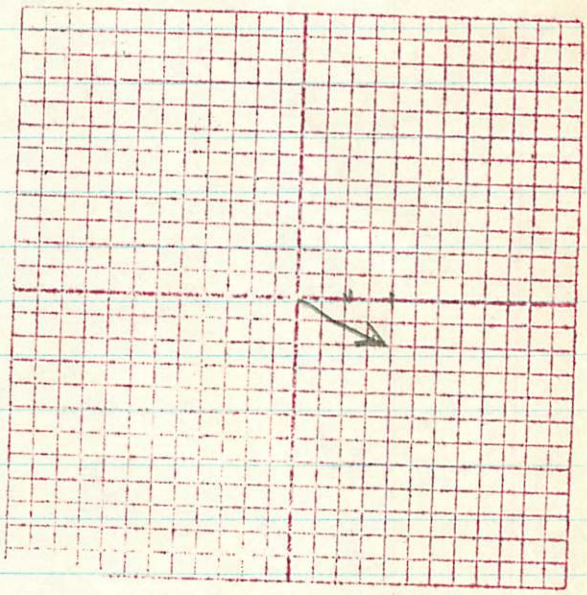


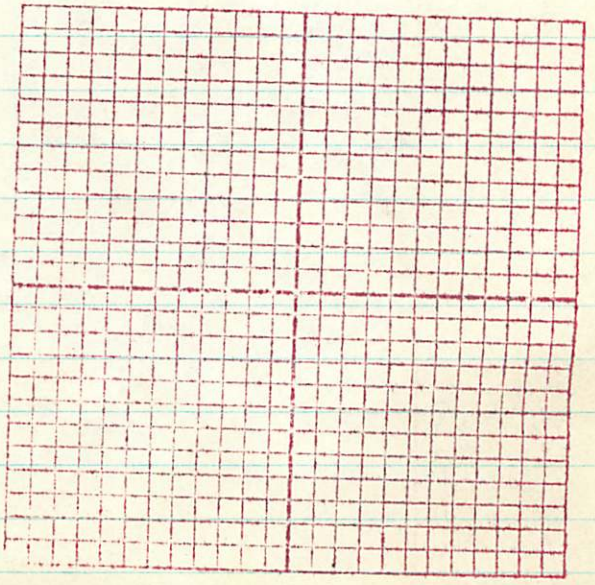
(22) $3\left[(\sqrt{3}, -1) + (\sqrt{2}, \sqrt{2})\right]$



(23) $(c_1 + c_2)\, v_1$

$(3 - 2)(\sqrt{3}, -1) = (\sqrt{3}, -1)$





Basis Vectors

Students should NOT follow along in text with this lecture.

Given: Vectors $\vec{v}_1, \vec{v}_2, \vec{a}$ & scalars c_1, c_2

Assume

① $\vec{a} = c_1\vec{v}_1 + c_2\vec{v}_2$

$\vec{a}, \vec{v}_1, \vec{v}_2$ are linearly dependent because we can solve for each of the three and the only operations will be add or subtraction of vectors. (Division & multiplication by scalar is basically the same.)

Let $\vec{v}_1 = (x_1, y_1)$ $\vec{v}_2 = (x_2, y_2)$ $\vec{a} = (x_3, y_3)$

Then ① above: $(x_3, y_3) = c_1(x_1, y_1) + c_2(x_2, y_2)$

$= (c_1x_1, c_1y_1) + (c_2x_2, c_2y_2)$

$(x_3, y_3) = (c_1x_1 + c_2x_2, c_1y_1 + c_2y_2)$

$\therefore$

$$\begin{cases} x_3 = c_1x_1 + c_2x_2 \\ y_3 = c_1y_1 + c_2y_2 \end{cases} \quad \& \quad y_3 = c_1y_1 + c_2y_2$$

$x_3y_1 = c_1x_1y_1 + c_2x_2y_1$

$y_3x_1 = c_1x_1y_1 + c_2x_1y_2$

$x_3y_1 - y_3x_1 = c_2(x_2y_1 - x_1y_2)$

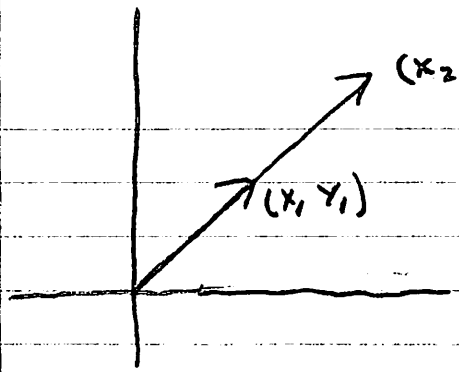
$$c_2 = \frac{x_3y_1 - y_3x_1}{x_2y_1 - x_1y_2} = \frac{x_1y_3 - x_3y_1}{x_1y_2 - x_2y_1}$$

c_1 can also be determined in the same manner.

Notice The Denominator: $x_1y_2 - x_2y_1 \neq 0$.

when would $x_1y_2 = x_2y_1$?

Consider ↓



Notice $x_2 = c \cdot x_1$ and $y = c\, y_2$
Because $\vec{v}_1$ and $\vec{v}_2$ are collinear-
then
$x_1 y_2 = y_2 y_1$ Becomes:

$$x_1 (c y_1) = (c x_1) y_1$$
$$c x_1 y_1 = c x_1 y_1$$

Thus the above work is valid providing that $\vec{v}_1$ and $\vec{v}_2$ are <u>NOT</u> COLLinear.

Summary thus far:

We can form any vector, $\vec{a}$, as a linear combination of two non collinear vectors, $\vec{v}_1$ & $\vec{v}_2$.

The meaning of this is that we can choose two convient vectors to work with then our only chore is to determine c_1, c_2 in order to express the given vector $\vec{a}$.

What are our two "convient" vectors?
lets call $\vec{i} = (1,0)$ and $\vec{j} = (0,1)$. These vectors are Perpendicular (called Orthogonal) These are known as our "Basis Vectors".
When we break up a single vector into Scalar times our Basis vectors it is called "resolving" the vector. an example where this would be used:
A pilot who knows is plane's air speed and angle of climb might need to know his climbing velocity (a vertical vector), and his ground velocity,

unit vectors.

we define $\frac{\vec{v}}{c} = \frac{1}{c}\vec{v}$. If you multiply any non zero vector by the reciprocal of it's norm we have "Normalized" the vector. a Normalized vector has length 1 and is in the same direction as the original vector.

Consider $v = (x, y)$ Then $\|v\| = \sqrt{x^2+y^2}$.

$$\frac{v}{\|v\|} = \frac{1}{\|v\|} v = \frac{1}{\sqrt{x^2+y^2}} (x, y) = \underbrace{\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)}$$

Our Normalized vector ↑

What is the Norm of this vector?

$$\sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2} = \sqrt{\frac{x^2+y^2}{x^2+y^2}} = 1$$

How do we determine the angle θ of a vector?

$$\cos\theta = \frac{x}{\|v\|} \text{ AND/OR } \sin\theta = \frac{y}{\|v\|}$$

in our original vector $\cos\theta_1 = \frac{x}{\sqrt{x^2+y^2}}$, in our Normalized vector $\cos\theta_2 = \frac{\frac{x}{\sqrt{x^2+y^2}}}{1} = \frac{x}{\sqrt{x^2+y^2}}$

2 possible values for θ yet.

ALSO!

$\sin\theta_1 = \frac{y}{\sqrt{x^2+y^2}}$ in original vector and in Normalized vector $\sin\theta_2 = \frac{\frac{y}{\sqrt{x^2+y^2}}}{1} = \frac{y}{\sqrt{x^2+y^2}}$. Thus, since

Both have the same we know the angle of our Normalized vector is the same as the original

eg (2) Find the unit vector v in the direction of $v_1 + v_2$ where $v_1 = \langle 3, -1 \rangle$ and $v_2 = \langle -6, 5 \rangle$

$v_1 + v_2 = \langle 3-6, -1+5 \rangle = \langle -3, 4 \rangle$

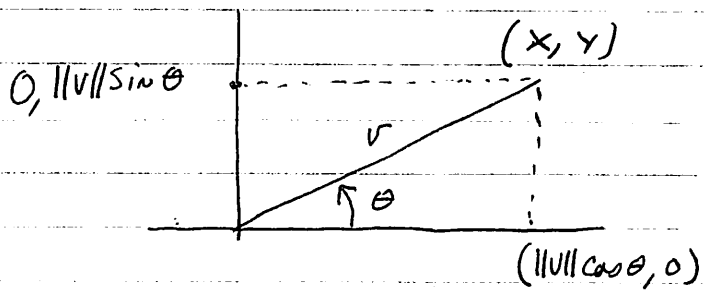
a unit vector, $\vec{u}$, of $\vec{v_3}$ is found by dividing the vector $\vec{v_3}$ by its norm. ie:

$$\vec{u} = \frac{\vec{v_3}}{\|\vec{v_3}\|}$$ for this problem we need the norm of $v_1 + v_2$ ie $\|v_1 + v_2\|$. So:

$$\sqrt{(-3)^2 + (4)^2} = 5 = \|v_1 + v_2\|.$$

$$\frac{\langle -3, 4 \rangle}{5} = \frac{1}{5}\langle -3, 4 \rangle = -\frac{3}{5}, \frac{4}{5}.$$

Given $\|v\|$ & θ Determine v.



$\vec{v} = (x, y) = \|v\| (\cos\theta, \sin\theta)$ OR

$\vec{v} = (x, y) = \|v\| [i\cos\theta + j\sin\theta]$

resolve the vector whose norm is 4 and whose direction angle is 30°.

$\vec{v} = 4\left[\frac{\sqrt{3}}{2}i + \frac{1}{2}j\right] = 2\sqrt{3}\,\vec{i} + 2\vec{j}$

Pg 221-222

$\vec{v}_1 = (-3, 2)$ $\vec{v}_2 = (5, 6)$, $\vec{v}_3 (3, -2)$ $c_1 = 2$ $c_2 = 1$ $c_3 = 3$

① $\vec{v}_1 + \vec{v}_2 = 2\vec{i} + 8\vec{j}$

⑤ $c_2\vec{v}_1 + c_1\vec{v}_3 = (1)(-3, 2) + 2(3, -2) = (-3, 2) + (6, -4) =$
$(3, -2) \Rightarrow 3\vec{i} - 2\vec{j}$

⑨ $(3, -2)$ $\sqrt{9+4} = \sqrt{13}$ $\therefore \left(\frac{3}{\sqrt{13}}, \frac{-2}{\sqrt{13}}\right)$

⑪ $2\vec{i} + 7\vec{j} \Rightarrow (2, 7) \Rightarrow \sqrt{4+49} = \sqrt{53}$ $\therefore$
$\left(\frac{2\sqrt{53}}{53}, \frac{7\sqrt{53}}{53}\right)$

⑮ $(6\cos 30°, 6\sin 30°) = 6(\cos 30°, \sin 30°)$

$6\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = (3\sqrt{3}, 3)$ $\sqrt{27+9} = \sqrt{36} = 6$

$\frac{6(\cos 30°, \sin 30°)}{6} = (\cos 30°, \sin 30°) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

⑱ $\|v\| = 5$ $\theta = 45$ $\therefore v = 5\left[\frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{j}\right]$

$v = \frac{5\sqrt{2}}{2}\vec{i} + \frac{5\sqrt{2}}{2}\vec{j}$

㉓ $\vec{i} - \vec{j} \Rightarrow \frac{\sqrt{3}-1}{2}\vec{i} - \frac{\sqrt{3}+1}{2}\vec{j}$

㉙ let $v = (x, y)$ then $cv = (cx, cy)$
$v + cv = (x + cx, y + cy) = (x(1+c), y(1+c))$
Let θ_1 = direction of v and θ_2 = direction of $v + cv$
WTS $\theta_1 = \theta_2$, ie same direction.

$\cos\theta_1 = \frac{x}{\sqrt{x^2+y^2}}$ $\sin\theta_1 = \frac{y}{\sqrt{x^2+y^2}}$

$\cos\theta_2 = \frac{x(1+c)}{\sqrt{x^2(1+c)^2 + y^2(1+c)^2}} = \frac{x(1+c)}{(1+c)\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$

$\sin\theta_2 = \frac{y(1+c)}{(1+c)\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$ since $\cos\theta_1 = \cos\theta_2$ & $\sin\theta_1 = \sin\theta_2$
$\theta_1 = \theta_2$.

$\tan\theta_1 = \frac{y}{x}$ $\cos\theta_2 = \frac{y(1+c)}{x(1+c)} = \frac{y}{x}$ But by itself is not enough.

Pg 221-222

(30) $\vec{V_1} = x_1\vec{i} + y_1\vec{j}$ $\quad \vec{V_2} = x_2\vec{i} + y_2\vec{j}$

Let θ_1 = direction of $\vec{V_1}$ & θ_2 = direction of $\vec{V_2}$

$$\cos\theta_1 = \frac{x_1}{\sqrt{x_1^2+y_1^2}} \qquad \cos\theta_2 = \frac{x_2}{\sqrt{x_2^2+y_2^2}}$$

$$\sin\theta_1 = \frac{y_1}{\sqrt{x_1^2+y_1^2}} \qquad \sin\theta_2 = \frac{y_2}{\sqrt{x_2^2+y_2^2}}$$

Also means that $\vec{V_1}$ & $\vec{V_2}$ are in different Quadrants,

if $V_1 \perp V_2$ then $\theta_1 = \theta_2 \pm 90 \Rightarrow$

$$\sin\theta_1 = \sin(\theta_2 \pm 90) = \sin\theta_2\cos 90 \pm \sin 90\cos\theta_2$$

$$\rightarrow \sin\theta_1 = \sin(\theta_2 \pm 90) = \pm\cos\theta_2$$

AND

$$\cos\theta_1 = \cos(\theta_2 \pm 90) = \cos\theta_2\cos 90 \mp \sin\theta_2\sin 90$$

$$\cos\theta_1 = \cos(\theta_2 \pm 90) = \mp\sin\theta_2$$

thus $\sin^2\theta_1 = \cos^2\theta_2$ and $\sin^2\theta_2 = \cos^2\theta_1$

$$\left(\frac{y_1}{\sqrt{x_1^2+y_1^2}}\right)^2 = \left(\frac{x_2}{\sqrt{x_2^2+y_2^2}}\right)^2 \text{ and } \left(\frac{x_1}{\sqrt{x_1^2+y_1^2}}\right)^2 = \left(\frac{-y_2}{\sqrt{x_2^2+y_2^2}}\right)^2$$

Squaring

$$\frac{y_1^2}{x_1^2+y_1^2} = \frac{x_2^2}{x_2^2+y_2^2} \quad \& \quad \frac{x_1^2}{x_1^2+y_1^2} = \frac{y_2^2}{x_2^2+y_2^2}$$

$$x_2^2y_1^2 + y_1^2y_2^2 = x_1^2x_2^2 + x_2^2y_1^2 \quad \& \quad x_1^2x_2^2 + x_1^2y_2^2 = x_1^2y_2^2 + y_1^2y_2^2$$

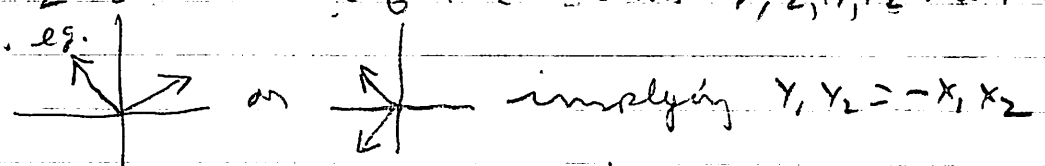
$$y_1^2y_2^2 = x_1^2x_2^2 \quad \& \quad x_1^2x_2^2 = y_1^2y_2^2$$

$$y_1y_2 = \pm x_1x_2 \quad \& \quad x_1x_2 = \pm y_1y_2$$

(I) (II)

either $y_1y_2 = x_1x_2$ or $y_1y_2 = -x_1x_2$

Since $V_1 \perp V_2$ at least one of the 4 items x_1, x_2, y_1, y_2 must be Negative. e.g.



or another

implying $y_1y_2 = -x_1x_2$

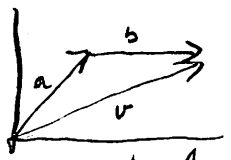
is the only possible relationship. Thus

$y_1y_2 = -x_1x_2 \Rightarrow x_1x_2 + y_1y_2 = 0.$

Students may follow in text. Pg 223–227

Dot Product.

Thus far we have considered the effect of a vector at a specific point.



The resultant of $\vec{a}$ & $\vec{b}$ is $\vec{v}$ that is for example a force, $\vec{b}$, applied at the head of $\vec{a}$.

If $\vec{b}$ moves along $\vec{a}$ exerting a constant force continuously then we have a different situation. The effect for us would be the "work done". i.e. How much work did these two vectors do?

This application is called the inner product, dot product or Scalar Product depending on the text you use.

If $v_1 = x_1\vec{i} + y_1\vec{j}$ and $v_2 = x_2\vec{i} + y_2\vec{j}$ then the inner product is given by:

$$\vec{v_1} \cdot \vec{v_2} = x_1x_2 + y_1y_2$$

↑ This is why it is also called Dot Product.

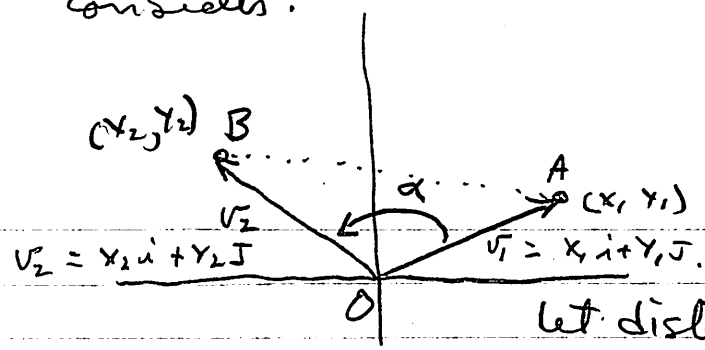
Notice that x_1, x_2, y_1, y_2 are $\in \mathbb{R}$ i.e. NOT Vectors. this is the reason this product is called Scalar Product. i.e. The product is a real #, a scalar.

If $v_1 = 2i - 5j$ & $v_2 = 3i - j$ then $v_1 \cdot v_2 =$

$$2 \cdot 3 + (-5)(-1) = 6 + 5 = 11$$

↓

Consider.



WANT To find α. (2)

If we draw $\overline{AB}$ we have a $\triangle AOB$, apply Law of Cos.

Let distance of $AB = d(AB)$

$$[d(AB)]^2 = \|v_1\|^2 + \|v_2\| - 2\|v_1\|\|v_2\| \cos\alpha \quad \text{Solve for } \cos\alpha.$$

$$\cos\alpha = \frac{\|v_1\|^2 + \|v_2\| - [d(AB)]^2}{2\|v_1\|\,\|v_2\|} \qquad \|v_1\| = \sqrt{x_1^2 + y_1^2} \text{ etc.}$$

$$\cos\alpha = \frac{x_1^2 + y_1^2 + x_2^2 + y_2^2 - \left[(x_2 - x_1)^2 + (y_2 - y_1)^2\right]}{2\left(\sqrt{x_1^2 + y_1^2}\right)\left(\sqrt{x_2^2 + y_2^2}\right)}$$

$$= \frac{x_1^2 + y_1^2 + x_2^2 + y_2^2 - \left[x_2^2 - 2x_1x_2 + x_1^2 + y_2^2 - 2y_1y_2 + y_1^2\right]}{2(\quad)(\quad)}$$

$$= \frac{2x_1x_2 + 2y_1y_2}{2(\quad)(\quad)} = \frac{x_1x_2 + y_1y_2}{\sqrt{x_1^2 + y_1^2}\,\sqrt{x_2^2 + y_2^2}}$$

$$\cos\alpha = \frac{v_1 \cdot v_2}{\|v_1\|\,\|v_2\|} \Rightarrow v_1 \cdot v_2 = \|v_1\|\,\|v_2\|\cos\alpha.$$

This ends the original P6lm. However noticed another expression for dot Product.

This expression is more useful since there is no division involved, it includes for eg zero vector.

Eg 2 $\|v_1\| = 7$ $\|v_2\| = 12$ $\theta_1 = 30°$ $\theta_2 = 150°$ $v_1 \cdot v_2 = ?$

$7 \cdot 12 \cdot \cos 120° = -42$. In application I do not know the significance of the Negative. (Yet.)

If neither v_1 nor v_2 is the zero vector but $v_1 \cdot v_2 = 0$ then $\cos\alpha = 0 \Rightarrow \alpha = 90$ i.e. $v_1 \perp v_2$ (v_1 & v_2 are orthogonal). Also note:

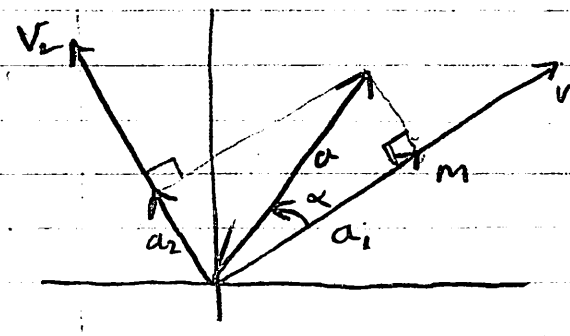
If v_1 or v_2 were zero vector then we can still say $v_1 \perp v_2$ since zero vector has any direction we consider convenient, so:

$v_1 \cdot v_2 = 0$ iff v_1 & v_2 are orthogonal vectors,

$v_1 \cdot v_2 = \pm \|v_1\| \|v_2\|$ iff v_1 & v_2 collinear.

$v_1 \perp v_2$

Given 3 vectors $\vec{v_1}, \vec{v_2}, \vec{a}$



a_1 is the Projection of a on v_1 and a_2 is the projection of a on v_2

By def $a = a_1 + a_2$

Therefore we have a in terms of a_1 and a_2 which are in the same direction as v_1 and v_2 respectively. Clearly a_1 and v_1 are collinear (as are a_2 and v_2) therefore $a_1 = c_1 v_1$ (a linear combination).

$\frac{a_1}{\|v_1\|} = \frac{c_1 v_1}{\|v_1\|} \Rightarrow \vec{a_1} = c_1 \|v_1\| \frac{v_1}{\|v_1\|}$ this implies that $c_1\|v_1\|$ is $\|a_1\|$. $c_1\|v_1\|$ (is a scalar). Times the unit vector $= \vec{a_1}$. Notice that we must consider

This $|c_1\|v_1\||$ rather than $c\|v_1\|$. Since distance is ≥ 0 and $c_1 \in \mathbb{R}$, in order to preserve not only the collinear property but the same direction as well. (It could happen that we have collinear vectors in opposite direction.

[[If $\vec{a}$ is between $\vec{v}_1$ & $\vec{v}_2$ as indicated in picture the Absolute value will NOT be needed]]

Consider the vector v_1 as though it were an axis, then you will see that via def of Cos FCTN

$$\cos\alpha = \frac{c_1\|v_1\|}{\|a\|} = \frac{\|a_1\|}{\|a\|} \quad \text{thus}$$

$\|a\|\cos\alpha = \|a_1\|$. Also, Consider $\cos\alpha$ in this form: $\vec{v}_1 \cdot \vec{v}_2 = \|v_1\|\|v_2\|\cos\alpha$

Let $a = v_2$ $v_1 = v_1$ $\alpha = \alpha$

$$\vec{v}_1 \cdot \vec{a} = \|v_1\|\|a\|\cos\alpha$$

$$\frac{\vec{v}_1 \cdot \vec{a}}{\|v_1\|} = \|a\|\cos\alpha \quad \text{So.} \quad \|a_1\| = \frac{\vec{v}_1 \cdot \vec{a}}{\|v_1\|} \ \bigstar$$

Also since $\frac{\vec{a}_1}{\|a_1\|} = \frac{\vec{v}_1}{\|v_1\|}$ (Both unit vectors & in the same direction)

then $\vec{a}_1 = \frac{\|a_1\|\vec{v}_1}{\|v_1\|} = \frac{\|a_1\|}{\|v_1\|}\vec{v}_1$ using $\bigstar$

$$\vec{a}_1 = \frac{\vec{v}_1 \cdot \vec{a}}{\|v_1\|}\frac{v_1}{\|v_1\|} = \frac{\vec{v}_1 \cdot \vec{a}}{\|v_1\|^2}v_1$$ clearly a_2 & v_2 would combine in the same manner to yield ↓

remember we have! $a = a_1 + a_2$

So, $\vec{a} = \frac{\vec{v_1} \cdot \vec{a}}{\|v_1\|^2} v_1 + \frac{\vec{v_2} \cdot \vec{a}}{\|v_2\|^2} v_2$

eg 3 $v_1 = \frac{\sqrt{3}}{2} i - \frac{1}{2} j$ $v_2 = \frac{1}{2} i + \frac{\sqrt{3}}{2} j$

find a linear combination of v_1 & v_2 = to

$a = \frac{\sqrt{3}}{2} i + \frac{1}{2} j$. Notice $v_1 \perp v_2$ $\|v_1\| = 1$, $\|v_2\| = 1$

$v_1 \cdot a = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$ $v_2 \cdot a = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

$a = \frac{1}{2} v_1 + \frac{\sqrt{3}}{2} v_2$

One possible use for this projection technique might be!

Given two existing forces v_1 & v_2 which are influencing v_3. You want to exert force a on v_3 so the adjustment to v_1 and v_2 is the adjustment which you require.

Pblms Students should Pg 228
Copy all 21-30 into notes.

(21)

↳ TP $\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_2 \cdot \vec{v}_1$

$\vec{v}_1 \cdot \vec{v}_2 = x_1 x_2 + y_1 y_2 = x_2 x_1 + y_2 y_1 = \vec{v}_1 \cdot \vec{v}_2$

(22) $c_1(\vec{v}_1 \cdot \vec{v}_2) = c_1 \vec{v}_1 \cdot \vec{v}_2$

$c_1(x_1 x_2 + y_1 y_2) = c_1(x_1 x_2) + c_1(y_1 y_2) = (c_1 x_1) x_2 + (c_1 y_1) y_2 =$

$(c_1 x_1 + c_1 y_1) \cdot (x_2 + y_2) = c_1(x_1 + y_1) \cdot (x_2 + y_2) = (c_1 \vec{v}_1) \cdot \vec{v}_2$

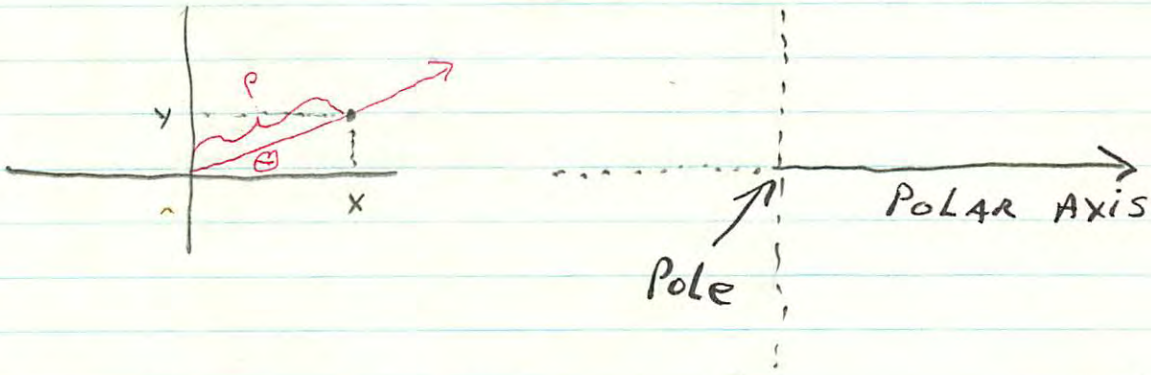
(23)

28 → 30.

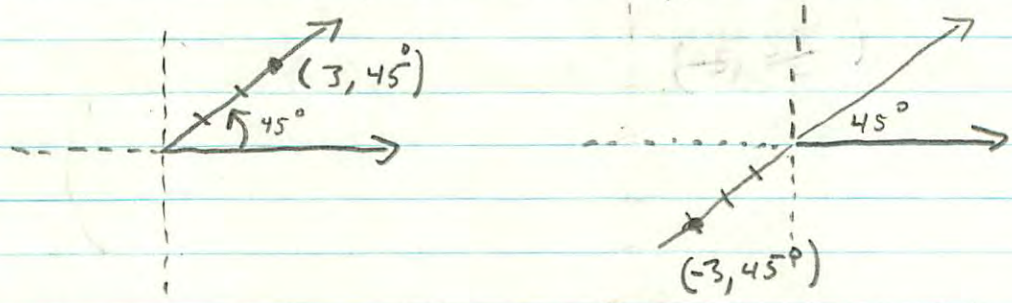
Polar Coordinates & Graphs

In chapt 3 we graphed equations involving the various trig fcts. on the standard x, y axes. Another way to graph these fcts would be in terms of θ, the angle in standard position, and the distance from the origin. We will call this distance r (rho). This system for graphing is called Polar coordinate System.

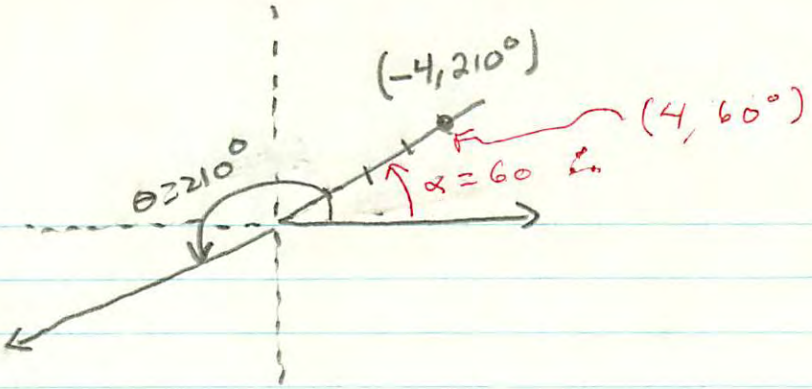
In cartesian coordinates, two numbers, (x, y) are used to measure the position of a point. Two numbers are also used in Polar Coordinate but these are two different numbers.



Coordinates in the Polar system are usually written (r, θ) where θ can be degrees or radians depending upon the text. Some texts reverse the order to (θ, r). This, you will see, has no effect on the graph.

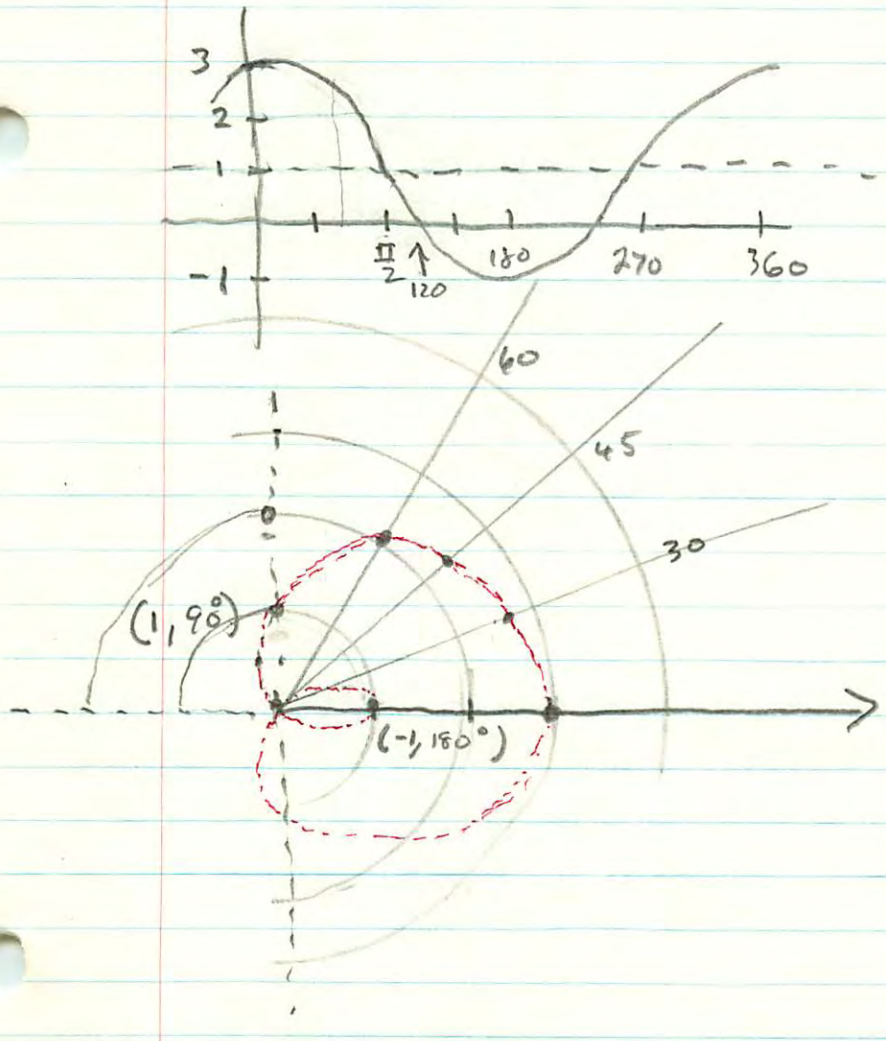


when $r < 0$ we measured the 3 units in the opposite direction of the Ray.



It should be clear that in the Polar System (unlike the Cartesian system) we have an infinite number of coordinates naming the same point.

Suppose we want to graph $r = 1 + 2\cos\theta$. remember what it looke like in chapt 3.



Picking some convient values.

(3, 0°)

(, 30°)

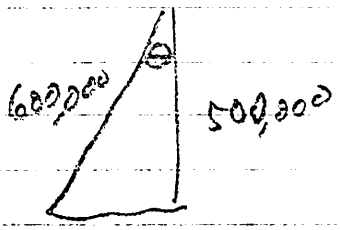
5

B) the circle is smaller because $\vec{h}$ is getting larger forcing the plane into a tighter circle.

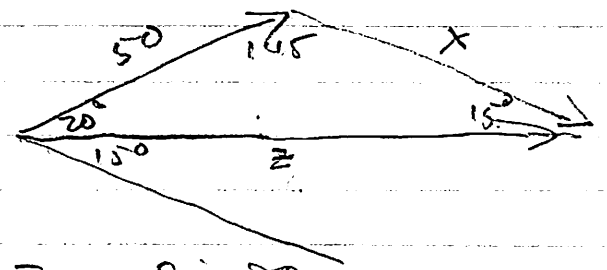
C) when a plane is not banking there is no horizontal force turning it.

D) $L = 600{,}000$

$\sec\theta = 1.200 \Rightarrow \theta \approx 33^\circ 30'$



⑥



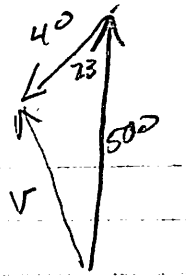
$\frac{\sin 15}{50} = \frac{\sin 20}{x}$ 1.32

$x = \frac{50 \sin 20}{\sin 15} = 50 \cdot \frac{.3420}{.2588} = \underline{66}$

$\sin 145 = \frac{z \sin 15}{50}$

$\frac{.5736}{.2588} = 2.22$ $\quad 50(2.22) = \underline{111.}$

④



a) $v^2 = 40^2 + 500^2 - 2(40)(500)\cos 23°$

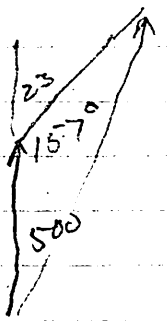
$v^2 = 1600 + 250{,}000 - 40000(.9205)$

$v^2 = 257{,}600 - 36820$

$v^2 = 214780$

$v \approx 463$ kph

b)

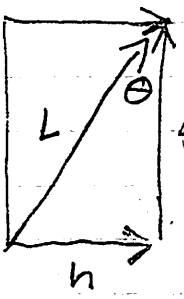


$v^2 = 251600 + 36820$

$v^2 = 288420$

$v = 537$

⑤



$\frac{L}{500{,}000} = \sec\theta$ and $\frac{h}{500000} = \tan\theta$

So

$L = 500{,}000 \sec\theta \quad h = 500{,}000 \tan\theta$

Ai $500{,}000 \sec 10° = 500{,}000(1.015) = 507713 = L$

$500{,}000 \tan 10° = 500{,}000(.1763) = 88163 = h$

ii $500{,}000 \sec 20° = 500{,}000(1.064) = 532088 = L$

$500{,}000 \tan 20 = 500{,}000(.3640) = 181985 = h$

iii $500{,}000 \sec 30 = 500{,}000(1.155) = 577350 = L$

$500{,}000 \tan 30 = 500{,}000(.5734) = 288675 = h$

iv $500{,}000 \sec 0° = 500{,}000 = L$

$500{,}000 \tan 0° = 0 = h$

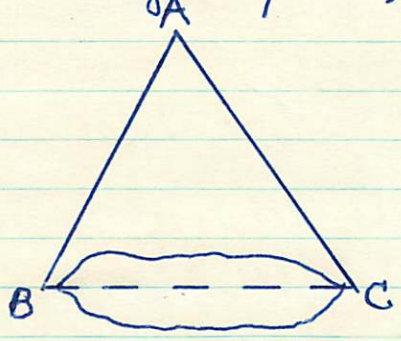
v undefined

1. Two forces are to act on a body to produce a resultant of 125 lb. If the lines of action of the two forces form an angle of 70°20′ and one of the forces is 79 lb., find, to the nearest 10 minutes, the angle which the other force forms with the resultant.

2. Two forces are to act on a body to produce a resultant of 85 pounds. If the lines of action of the two forces form an angle of 62° and one of the forces is 52 pounds, find, to the nearest pound, the other force.

3. The diagonals of a parallelogram make an angle of 28°20′ with each other. The shorter diagonal is 38.6 in. and the shorter side is 20.4 in. Find the other diagonal, to the nearest tenth of an inch.

4. In order to measure the distance BC across a swamp, a surveyor measures $\overline{BA}$, $\overline{AC}$, and $\angle CBA$. If $\angle CBA = 48°10'$, the length of $\overline{AC} = 4840$ ft., and the length of $\overline{BA} = 3470$ ft., find the length of $\overline{BC}$, to the nearest 10 feet.



5. Two forces of 437 pounds and 876 pounds, respectively, act upon a body at an acute angle with each other. The angle between the resultant force and the 437-pound force is 41°10'. Find, to the nearest ten minutes, the angle formed by the 437-pound and the 876-pound forces.

6. In triangle ABC, AB=40, BC=34, and B=50°. Find, to the nearest integer, the length of AC.

7. In order to find the distance between two points, A and B, separated by a swamp, a station C was chosen and the distance CA and CB were found to be 350 yards and 380 yards respectively. Angle ACB was found to be 62°. Find, to the nearest yard, the distance from A to B.

8. Two forces of 38 pounds and 27 pounds are acting on a body at an angle of 48°. Find the resultant, correct to the nearest pound.

9. Two forces of 55 pounds and 44 pounds are acting on a body at an angle of 64°20'. Find the resultant, correct to the nearest pound.

10. The distance from an observer's eye to the summits of two hills are 10 miles and 28 miles. If the angle between his lines of sight is 104°, find, to the nearest mile, the shortest distance from one summit to the other.

16. Two observers 5280 feet apart on a straight horizontal road observe a balloon between them directly above the road. At the points of observation, the angles of elevation of the balloon are 60° and 75°. Find, correct to the nearest foot, the height of the balloon.

17. A sign 50.7 feet high is put on top of a building. From a point on the ground the angles of elevation of the top and bottom of the sign are 42° 30' and 35° 20'. Find the height of the building, correct to the nearest foot.

18. A tree is growing at the bank of a river. The angle of elevation of the top from a point directly across on the other bank is 37° 50'. From a second point on the same bank, 125 feet from the first and in line with the first point and the tree, the angle of elevation of the top of the tree is 22° 30'. Find, to the nearest foot, the width of the river.

19. From the top of a house 32 feet high, the angle of elevation of the top of a pole is 12° 10'. At the ground level of the house, the angle is 40° 50'. Find, correct to the nearest foot, the height of the pole.

11. The diagonals of a parallelogram are 60 and 70 inches and intersect at an angle of 68°. Find the shorter side of the parallelogram, correct to the nearest inch.

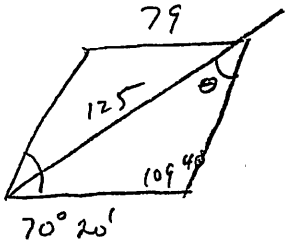
12. A triangular plot of ground measures 18 rods on one side and 24 rods on another side. The angle included between these sides is 72°. Find, to the nearest rod, the amount of fencing required to enclose the plot.

13. Two sides of a triangular plot of ground whose lengths are 90 feet and 110 feet form an angle of 103°20'. Find, to the nearest foot, the perimeter of the plot.

14. From two points on level ground in line with the foot of a tree and on the same side of the tree, the angles of elevation of the top of the tree are 41° and 47°. If the distance between the two points is 50 feet, find, correct to the nearest foot, the height of the tree.

15. A vertical tower stands at the top of a hill which is inclined 16° to the horizontal. At a point 95 feet down the hill from the base of the tower, the tower subtends an angle of 38°. Find, correct to the nearest foot, the height of the tower.

Ditto

① 

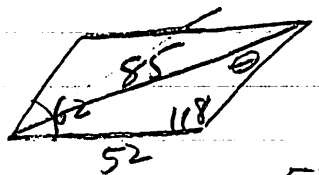
17960
7020
10940

8960
1940
70°20'

$\frac{\sin 109^\circ 40'}{125} = \frac{\sin \theta^\circ}{79} = \frac{\sin 70^\circ 20'}{125}$

Consecutive $\angle$s of ||gram are supplementary

$79 \cdot \frac{.19417}{125} = \sin\theta = .15951 \Rightarrow \theta = 36^\circ 50'$

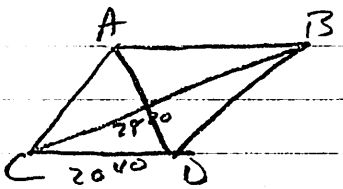
② 

90
−28
62

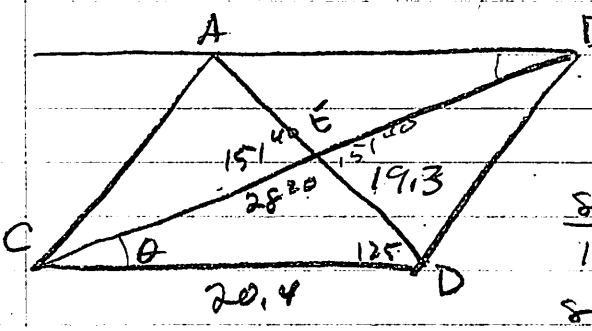
$\frac{\sin\theta}{52} = \frac{\sin 118}{85} = \frac{\sin 62}{85}$

$52 \frac{.18829}{85} = \sin\theta = .5402 \Rightarrow$

$\theta = 32^\circ 42'$

③ 

AD = 38.6 CD = 20.4



17960
2820
15140

Diags of ||gram bisect each other.

$\frac{\sin\theta}{19.3} = \frac{\sin 28^\circ 20'}{20.4}$

$\sin\theta = \frac{(.4746)(19.3)}{20.4} = .4490$

$\theta = 26^\circ 40'$

28
27
55

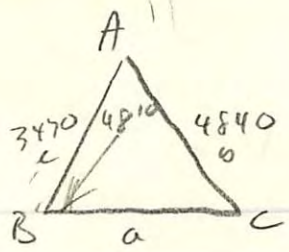
180
55
125

$\frac{\sin 26^\circ 40'}{19.3} = \frac{\sin 125}{CE}$

$CE = \frac{19.3 \sin 125}{.4488} = 35.2$

$CB = 70.4$

4.



$a^2 = c^2 + b^2 - 2ac\cos 48^{10}$

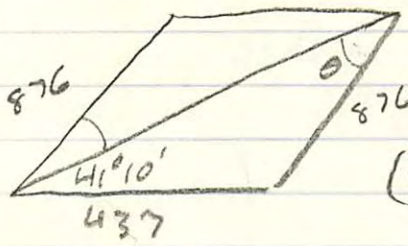
$a^2 = 12040900 + 23426600 - 47507400 - (33589600)(.6670)$

$47507400 - 22404263.2$

$a^2 = 25103136.8$

$a = 5010.3$ ~~6404~~

5.



$\frac{\sin 41^{10}}{876} = \frac{\sin\theta}{437}$

$(437)\frac{.6583}{876} = \sin\theta = .3284$

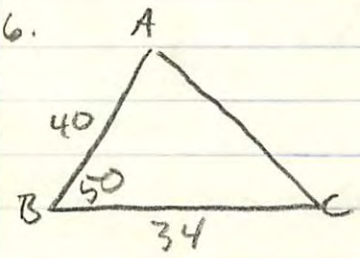
$\theta = 19^{0}10'$

$41^{0}10$

$60^{0}20'$

∡ Between 2 faces is $60^{0}20'$

6.



$b^2 = a^2 + c^2 - 2ac\cos B$

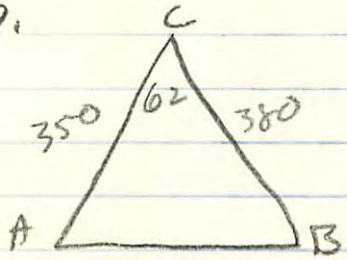
$b^2 = 1156 + 1600 - (2720)(.1428)$

$2756 \quad -1748.4$

$b^2 = 1007.6$

$b = 31.7 \Rightarrow 32$

7.



$c^2 = a^2 + b^2 - 2ab\cos C$

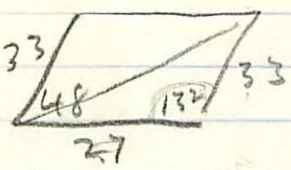
$c^2 = 144400 + 122500 - (266000)(.4695)$

$266900 - 124879.4$

$c^2 = 142020.6$

$c = 377$ yds.

8.



$c^2 = a^2 + b^2 - 2ab\cos C$

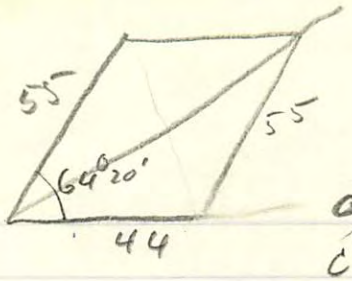
$c^2 = 729 + 1089 - (1782)(.6691)$

$1818 + 1192.39$

c^2 625.6

$c = 25.$

9.



$$\begin{array}{r} 18960 \\ 6426 \\ \hline 115^{\circ}40' \end{array}$$

$c^2 = a^2 + b^2 - 2ab\cos C$

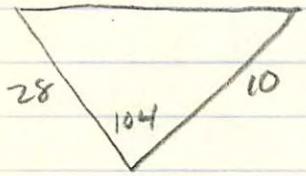
$c^2 = 1936 + 3025 - 4840(-.4331)$

$4961 + 2096$

$c^2 = 7057$

$c = 84$

10.



$c^2 = 784 + 100 - 280\cos 106^{\circ}$

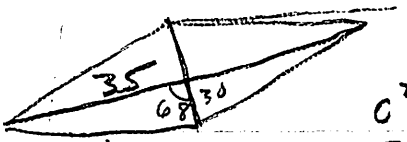
$= 884 - 280(-.2756)$

$884 + 77.2$

$= 961$

$c \approx 31$

961
77
.275

11 

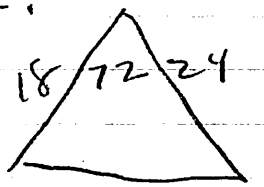
$c^2 = a^2 + b^2 - 2ab\cos C$

$c^2 = 900 + 1225 - (2100)(.3746)$

$2125 - 786.7$

$c^2 = 1338$

$= 36.6$

12. 

$c^2 = 18^2 + 24^2 - 18 \cdot 24 \cdot 2 \cos 72$

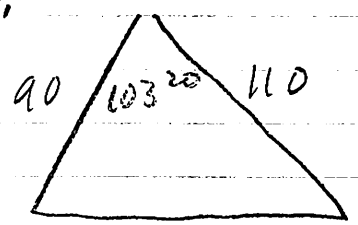
$324 + 576 - (864)(.3090)$

$900 - 267$

$c^2 = 633$

$c = 25$

67 Rods of fencing total

13, 

$c^2 = 90^2 + 110^2 - 90 \cdot 110 \cdot 2 \cdot \cos 103^{20}$

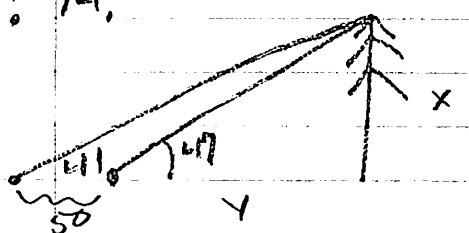
$8100 + 12100$

$20200 - (19800)(-.2300)$

4555.0

24755

$c = 157.3$

? 14, 

$\tan 47 = \frac{x}{y}$ $\quad y = \frac{x}{\tan 47}$

$\tan 41 = \frac{x}{y+50}$ $\quad y = \frac{x}{\tan 41} - 50$

$\frac{x}{\tan 47} = \frac{x}{\tan 41} - 50$

$x\left(\frac{1}{\tan 47} - \frac{1}{\tan 41}\right) = -50$

$.9325 - 1.1504$

$x = \frac{-50}{-.2179}$

$x = 230'$

Pg 265-268

Read pgs 265-267 NO lecture required.
HW routine

1, -5, 7, 9, 13, 15, -17

Pg 268

(15) $2 - xi + 3x = 16y + yi - 12i$

$2 + 3x - xi = 16y + (y-12)i$

$2 + 3x = 16y$ $\quad -x = y - 12$

$x = 12 - y$

$2 + 3(12-y) = 16y$

$2 + 36 - 3y = 16y$

$38 = 19y$

$y = 2 \quad \therefore \quad x = 10 \quad (10, 2)$

(10, 2)

(16) $x - 7 + 3xi = 3yi - 2y - 6i$

$(x-7) + 3xi = -2y + (3y-6)i$

$x - 7 = -2y$ $\quad 3x = 3y - 6$

$x - 2 - 7 = -2y$ $\quad x = y - 2$

$3y = 9$

$y = 3 \quad \therefore \quad x = 1 \quad (1, 3)$

(17) $x - 7 + 3xi$

etc

(21) $2\cos x + 2i\sin x = 1 + \sqrt{3}i$

$2\cos x = 1 \quad 2\sin x = \sqrt{3}$

$\cos x = \frac{1}{2} \quad \sin x = \frac{\sqrt{3}}{2} \quad \therefore x = \frac{\pi}{3}$

(22) $\sqrt{2}\cos x + \sqrt{2}i\sin x = 1 + i$

$\sqrt{2}\cos x = 1 \quad \sin x = \frac{\sqrt{2}}{2} \quad \therefore x = \frac{\pi}{4}$

$\cos x = \frac{\sqrt{2}}{2}$

(23) $4\cos x + 4i\sin x = -2 + 2\sqrt{3}i$

$\cos x = \frac{-1}{2} \quad \sin x = \frac{\sqrt{3}}{2} \quad x = \frac{2\pi}{3}$

(24) $3\cos x + 7i\sin x = -3i$

$\cos x = -1 \quad \sin x = 0 \quad \therefore x = \pi$

(25) $(\cos x + i\sin x)^2 = 2i$

$\cos^2 x - \sin^2 x + (2\sin x \cos x)i \qquad x \in \{\frac{\pi}{4}, \frac{5\pi}{4}\}$

$\cos^2 x = \sin^2 x \quad \sin x \cos x = 2$

(26) $(\cos x + i\sin x)^2 = (\cos x - i\sin x)^2$

$\cos^2 x - \sin^2 x + 2i\sin x\cos x = \cos^2 x - \sin^2 - 2i\sin x\cos x$

$\cos^2 x - \sin^2 x = \cos^2 x - \sin^2 x \quad 2\sin x\cos x = -2\sin x\cos x$

$= $ identity $\quad \sin x\cos x = 0$

$x \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$

(27) Show $z + \bar{z} \in \mathbb{R}$

$a+bi + a-bi = 2a$ since $a \in \mathbb{R}$ $2a \in \mathbb{R}$,

(28) Show $z - \bar{z} \in$ imag.

$a+bi - (a-bi) = 2bi \therefore$ imag, #

(29) WTP $z\bar{z} \in \mathbb{R}$.

$$(a+bi)(a-bi) = a^2+b^2+0i \therefore \in \mathbb{R}$$

(30) $\bar{z}_1 + \bar{z}_2 = \overline{z_1+z_2}$

let $z_1 = a+bi \quad z_2 = c+di$

$$a-bi + c-di = (a+c)-(b+d)i = \overline{(a+c)+(b+d)i}$$

(31) $\bar{z}_1\bar{z}_2 = \overline{z_1 z_2}$

$$(A-bi)(C-di) = (AC-bc)-(bc+Ad)i = \overline{(a+bi)(c+di)}$$

(32) WTP $\dfrac{\bar{z}_1}{\bar{z}_2} = \overline{\left(\dfrac{z_1}{z_2}\right)}$